Gauge fixing and the Hamiltonian for cylindrical spacetimes

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We introduce a complete gauge fixing for cylindrical spacetimes in vacuo that, in principle, do not contain the axis of symmetry. By cylindrically symmetric we understand spacetimes that possess two commuting spacelike Killing vectors, one of them rotational and the other one translational. The result of our gauge fixing is a constraint-free model whose phase space has four field-like degrees of freedom and that depends on three constant parameters. Two of these constants determine the global angular momentum and the linear momentum in the axis direction, while the third parameter is related with the behavior of the metric around the axis. We derive the explicit expression of the metric in terms of the physical degrees of freedom, calculate the reduced equations of motion and obtain the Hamiltonian that generates the reduced dynamics. We also find upper and lower bounds for this reduced Hamiltonian that provides the energy per unit length contained in the system. In addition, we show that the reduced formalism constructed is well defined and consistent at least when the linear momentum in the axis direction vanishes. Furthermore, in that case we prove that there exists an infinite number of solutions in which all physical fields are constant both in the surroundings of the axis and at sufficiently large distances from it. If the global angular momentum is different from zero, the isometry group of these solutions is generally not orthogonally transitive. Such solutions generalize the metric of a spinning cosmic string in the region where no closed timelike curves are present.

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I. INTRODUCTION

Vacuum cylindrical spacetimes have received intensive study in general relativity. The reasons for this interest can be found in the fact that cylindrical symmetry allows for a rich variety of physical phenomena while considerably simplifying Einstein's equations, so that one can obtain non-trivial exact solutions [1]. The first family of time-dependent cylindrical spacetimes in vacuo seems to have been found by Beck in the 1920's [2]. This family was rediscovered ten years later by Einstein and Rosen, in a systematic investigation of all cylindrically symmetric solutions that describe linearly polarized radiation [3]. The most general solution corresponding to cylindrical gravitational waves in vacuo (without the condition of linear polarization) was analyzed by Ehlers and collaborators, and independently by Kompaneets [4]. By studying the dynamical equations, Thorne [5] succeeded in constructing a covariant vector that satisfies a conservation law and provides a notion of energy for these cylindrical spacetimes. This C energy, which is positive and localizable, is in fact an energy density per unit length along the axis of symmetry. In the 1970's, Kuchař discussed the canonical formalism for Einstein-Rosen waves in the context of quantum gravity [6]. More recently, cylindrical gravitational waves have been considered as a particular case of spacetimes that possess a translational spacelike Killing field [7,8]. This class of spacetimes can be reduced to three dimensions using their symmetry. In this way, Ashtekar and Varadarajan showed that cylindrical waves admit a well-defined Hamiltonian formalism and that the Hamiltonian that generates asymptotic time tanslations at spatial infinity is not exactly the (total) C energy, but a non-polynomial function of it which, in addition to being positive, turns out to be bounded also from above [7,9]. The same conclusion about the value and bounds of the Hamiltonian was obtained from a purely four-dimensional perspective by Romano and Torre [10] and, for the particular case of Einstein-Rosen waves, also by analyzing the asymptotic structure at null infinity of the three-dimensional Killing reduction of the system [8].

The wave solutions in vacuo analyzed in all these works are obtained by adopting a definition of cylindrical symmetry that might be considered too restrictive. In addition to the existence of a translational and a rotational Killing field, it is assumed that the spacetime manifold contains at least part of the axis of cylindrical symmetry, namely, the set of fixed points of the axial Killing field [11]. Under such hypotheses, the geometry must be regular at the axis, and it is then possible to show that the isometry group generated by the two Killing vectors is Abelian [11] and orthogonally transitive [12–14]; i.e., the Killing orbits admit orthogonal surfaces. Obviously, the assumption of regularity eliminates interesting possibilities that have found applications in astrophysics and cosmology. This is the case, e.g., of straight cosmic strings, namely, one-dimensional topological defects with a linear energy density that introduce a conical singularity at the axis and, therefore, a deficit angle in the geometry [15]. Orthogonal transitivity (a consequence of the regularity at the axis) precludes as well the existence of a global rotation [13] which is present, for instance, in spinning

string solutions [16,17]. These solutions have axial singularities produced by string-like defects that carry a nonzero angular momentum per unit length in the axis direction and, in principle, may have vanishing energy density. In the absence of gravitational radiation, the energy content and angular momentum due to a cosmic string were analyzed from a three-dimensional viewpoint by Deser, Jackiw, and 't Hooft [16] and also by Henneaux [18]. On the other hand, a proposal has been recently made to extend the concept of energy from Einstein-Rosen waves to orthogonally transitive spacetimes that contain a nonspinning cosmic string [19]. This proposal, nevertheless, is not based on a Hamiltonian analysis.

The purpose of the present work is to generalize the study of the Hamiltonian structure and physical degrees of freedom of vacuum cylindrical solutions to the case in which the axis of symmetry is not included in the spacetime, so that singularities can appear there. In more detail, we will assume that there exists an Abelian two-dimensional group of isometries, generated by an axial and a translational spacelike Killing field, but we will not suppose that the axis belongs to the vacuum spacetime or that the isometry group is orthogonally transitive. Our aim is to introduce a complete gauge-fixing procedure and analyze the dynamics of the resulting reduced system. We want to investigate whether such a reduced dynamics admits a well-defined Hamiltonian formalism and, if the answer is in the affirmative, determine whether the existence of upper and lower bounds for the Hamiltonian still holds when the assumption of regularity at the axis is dropped.

The rest of the paper is organized as follows. In Sec. II, we develop a complete gauge fixing for cylindrical spacetimes in vacuo. For the momentum constraints that correspond to the Killing fields, the gauge freedom is fixed in Sec. II A. Section II B introduces a convenient change of metric variables, suitable for the study of cylindrical spacetimes. The gauge freedom associated with the remaining momentum constraint is removed in Sec. II C. Finally, we eliminate the Hamiltonian constraint in Sec. II D. The reduced system attained in this way is analyzed in Sec. III. Using the symplectic structure induced from general relativity, we find in Sec. III A a Hamiltonian that, at least formally, generates the dynamics of the reduced model. The explicit expression of the line element in terms of the four field-like degrees of freedom of the phase space of the system is presented in Sec. III B. We also include there the dynamical equations that dictate the evolution of the reduced model. In Sec. IV we prove that, when the reduced Hamiltonian is well defined, its range is bounded both from below and above. In Sec. V we discuss the conditions that ensure that the reduced Hamiltonian formalism is mathematically consistent and study the possible divergences of the metric at the axis. Section VI summarizes the main results of the work and includes some further comments. Finally, boundary conditions on our physical fields leading to an acceptable Hamiltonian formalism are presented in the Appendix.

II. GAUGE FIXING

Our starting point is the Hamiltonian formulation of general relativity. We assume that the spacetime is globally hyperbolic, so that it admits a 3+1 decomposition in sections of constant time t. In addition, we suppose that there exist two commuting spacelike Killing vector fields, one of them axial and the other one translational. Since the isometry group is Abelian (with non-null orbits), it is possible to choose spatial coordinates $x^i = \{z, \theta, u\}$ $(i = 1, 2, 3)$ such that ∂_z and ∂_θ are the translational and rotational Killing fields, respectively, and the spacetime metric is independent of z and θ [1,13]. As a consequence of this independence, the integral $\int dz \oint d\theta$ appears as a global factor in the gravitational action and in the symplectic structure of general relativity. We absorb the numerical value of $\int dz$ in Newton's constant (by renormalization if z has infinite length [20]) and call G the effective gravitational constant obtained in this manner. In addition, we normalize the coordinate θ so that it belongs to the unit circle S^1 (hence, $\oint d\theta = 2\pi$) and adopt units such that $4G = c = 1$. As for the spatial coordinate u, we choose its domain of definition equal to the real line. This choice is always compatible with the fact that ∂_θ is rotational if one accepts that the axis of symmetry is not included in our spacetime (think, e.g., of the change $u = \ln r$ if r is a radial coordinate).

Our system has the symplectic form

$$\Omega = \int_{I\!R} du\, \mathbf{d}\Pi^{ij} \wedge \mathbf{d}h_{ij}, \tag{2.1}$$

where $\mathbf{d}$ and $\wedge$ denote the exterior derivative and product. In terms of the induced metric h_{ij} and its extrinsic curvature K_{ij}, the canonical momenta can be written [21]

$$\Pi^{ij} = \frac{1}{2} h^{1/2} (h^{ik} h^{jl} - h^{ij} h^{kl}) K_{kl}, \tag{2.2}$$

with h and h^{ij} being the determinant and the inverse of the three-metric h_{ij}. The non-vanishing Poisson brackets derived from the above symplectic form are

$$\{h_{ij}(u), \Pi^{kl}(\bar{u})\} = \delta_i^{(k} \delta_j^{l)} \delta(u - \bar{u}). \tag{2.3}$$

Here, δ^i_j and $\delta(u)$ are the Kronecker delta and the Dirac delta, and the indices in parentheses are symmetrized. Calling $\tilde{\mathcal{H}}$ the densitized Hamiltonian constraint (i.e., the product of the Hamiltonian constraint by $h^{1/2}$) and $\mathcal{H}_i$ the momentum constraints, the time derivative of any function F on phase space is then given by the formula

$$\dot{F} = \partial_t F + \left\{ F, \int_{I\!R} du (\underset{\sim}{N} \tilde{\mathcal{H}} + N^i \mathcal{H}_i) \right\}, \tag{2.4}$$

where the overdot denotes the time derivative, ∂_t is the partial derivative with respect to the explicit time dependence, N^i is the shift vector, and $\underset{\sim}{N} = h^{-1/2} N$ is the densitized lapse (N being the lapse function) [21].

A. Momentum constraints for the Killing fields

Let us first fix the gauge freedom associated with the momentum constraints of the two coordinates $x^a = \{z, \theta\}$ ($a, b = 1, 2$ from now on). Remembering the independence of the metric on these coordinates, one can check [13,20] that, in our system of units,

$$\mathcal{H}_a = -2(h_{ai}\Pi^{iu})' . \qquad (2.5)$$

Here, the prime denotes the derivative with respect to u and we have introduced the alternative notation u for the spatial index $i = 3$. It is then possible to remove the corresponding gauge freedom by demanding that, when restricted to the sections of constant time (and only then), the action of the isometry group be orthogonally transitive, namely, that $h_{au} = 0$. It is easily seen that these gauge conditions are second class with the momentum constraints that we want to eliminate [20]. In order to arrive at a consistent gauge fixing, we therefore must only find values for the shift components N^a such that the conditions $h_{au} = 0$ are stable in the evolution. Using the fact that the solution to the momentum constraints $\mathcal{H}_a = 0$ is given in our gauge by $\Pi^{au} = h^{ab}c_b/4$, with $c_a(t)$ being two real functions of the time coordinate, a trivial calculation leads to the conclusion that the stability condition $\dot{h}_{au} = 0$ implies

$$N^a(u) = \int_u^\infty \underset{\sim}{N} h_{uu} h^{ab} c_b. \qquad (2.6)$$

Two additive (time-dependent) integration constants have been removed from N^a by imposing that these components of the shift vanish in the limit $u \to \infty$ or, equivalently, by a suitable redefinition of the coordinates x^a.

On the other hand, employing Eq. (2.4), it is possible to check that the dynamical evolution leaves invariant the variables Π^u_a, so that the functions $c_a = 4\Pi^u_a$ are in fact constants. Furthermore, using relation (2.2), one can show that $c_a = |g|^{-1/2}\tilde{\eta}^{\gamma\mu\nu\sigma}\, {}^{(1)}X_\mu\, {}^{(2)}X_\nu\, {}^{(a)}X_{\sigma;\gamma}$ [13,22], where g is the determinant of the four-metric, the semicolon stands for covariant derivative, Greek letters denote spacetime indices, $\tilde{\eta}^{\gamma\mu\nu\sigma}$ is the totally antisymmetric Levi-Cività tensor density, and ${}^{(a)}X$ are the two Killing fields, ∂_z ($a = 1$) and ∂_θ ($a = 2$). It therefore follows that the orbits of these Killing fields admit orthogonal surfaces if and only if $c_a = 0$ [1,12]. It is clear that the constants c_a are intimately related to global properties of the spacetime. Whenever they are different from zero, the geometry cannot be regular at the axis and the sections of constant time of the vacuum spacetime cannot have the topology of $I\!R^3$. Since we are not assuming orthogonal transitivity, we will not impose that c_a vanish. Nevertheless, although we will allow for the presence of non-zero constants c_a in our calculations, they will be treated as parameters that determine different sectors of the cylindrical reduction of general relativity, and not as physical degrees of freedom of the theory.

B. Change of metric variables

After the above partial gauge fixing, we will introduce a change of metric variables that leads to a much more convenient expression for the line element of our spacetimes with two commuting Killing fields, namely,

$$ds^2 = e^{2w+y}\left[-f^2\underset{\sim}{N}^2dt^2 + (du + N^u dt)^2\right] + e^y f^2$$
$$\times(d\theta + N^\theta dt)^2 + e^{-y}\left[dz - v d\theta + (N^z - vN^\theta)dt\right]^2 . \qquad (2.7)$$

The new metric variables that replace h_{uu} and the symmetric two-metric h_{ab} are $q \equiv \{f, v, y, w\}$. The restriction to (inequivalent) positive definite three-metrics h_{ij} requires that f be (e.g.) strictly positive whereas the rest of metric variables must be real. The momenta p_q canonically conjugate to the metric variables q can be easily found: $p_q = \Pi^{uu}\partial_q h_{uu} + \Pi^{ab}\partial_q h_{ab}$. Then, the reduced symplectic form of our gauge-fixed model is $\Omega_1 = \int du \mathbf{d}p_q \wedge \mathbf{d}q$. On the other hand, the two constraints that remain on the system can be written

$$\tilde{\mathcal{H}} = \frac{(y'f)^2}{4} + \frac{(v')^2}{4}e^{-2y} + p_y^2 + p_v^2 f^2 e^{2y} - f p_w p_f$$
$$+f(f'' - f'w') + \frac{e^{2w}}{4f^2}\left[(c_\theta + c_z v)^2 + c_z^2 f^2 e^{2y}\right] . \qquad (2.8)$$

$$\mathcal{H}_u = -p_w' + p_f f' + p_v v' + p_w w' + p_y y'. \qquad (2.9)$$

Notice that, when $c_z = c_\theta = 0$, these formulas reproduce the results obtained for spacetimes that are regular at the axis [10,23,24]. Finally, it is possible to check that the equations of motion obtained from Eq. (2.4) for the degrees of freedom of our gauge-fixed model coincide in fact with those generated in our reduced system by the Hamiltonian $\int du(\underset{\sim}{N}\tilde{\mathcal{H}} + N^u\mathcal{H}_u)$.

C. Radial momentum constraint

Our next step in the process of gauge fixing consists in eliminating the gauge freedom associated with the momentum constraint $\mathcal{H}_u$. This can be done, e.g., by imposing that the metric variable f be a fixed, strictly increasing function of only the coordinate u, so that, once the value of f is known, the coordinate u is totally determined. We note that, from expression (2.7), f^2 is just the determinant of the metric on Killing orbits. In particular, this metric degenerates if f vanishes. The set of points where $f = 0$, which are in principle excluded from our spacetime, would then correspond to the axis of symmetry. Thus, by introducing a change of coordinates that replaced u with f, we could interpret f as a kind of radial coordinate (recall that f is positive). We will return to this point in Sec. III B.

Let us hence impose the condition $f = r(u)$, where $r(u)$ is a fixed function that is strictly positive and increasing, so that $r(u) > 0$ and $r'(u) > 0$ everywhere. Although

the expression of $r(u)$ is given once and for all, we will not specify it explicitly; instead, we will treat $r(u)$ as an abstract fixed function. It is clear that our gauge-fixing condition does not commute under Poisson brackets with the constraint $\mathcal{H}_u = 0$. So our gauge fixing will be acceptable if we can find a value for the shift component N^u such that our choice of gauge is stable, namely, such that $\dot{f} = 0$ on our gauge section. One can see [20] that this requirement implies that $N^u = \underset{\sim}{N} p_w/(\ln r)'$. On the other hand, substituting $f = r(u)$ in Eq. (2.9), it is easy to find the expression for p_f that solves the constraint $\mathcal{H}_u = 0$. After removing the degrees of freedom f and p_f from the system, we arrive at a reduced phase space with symplectic form

$$\Omega_2 = \int_{I\!R} du\,(\mathbf{d}p_v \wedge \mathbf{d}v + \mathbf{d}p_w \wedge \mathbf{d}w + \mathbf{d}p_y \wedge \mathbf{d}y). \quad (2.10)$$

The system has only one constraint, the densitized Hamiltonian constraint (2.8) evaluated on our gauge section, which we will also call $\tilde{\mathcal{H}}$. One can check that the smeared constraint $\int du \underset{\sim}{N} \tilde{\mathcal{H}}$ generates the reduced dynamics via the Poisson brackets obtained from Ω_2.

D. Hamiltonian constraint

In order to complete our gauge fixing, we must remove the gauge freedom corresponding to the densitized Hamiltonian constraint. One can use this freedom to impose that the metric induced on the reference surfaces with coordinates t and u be diagonal. Since $g_{tu} = g_{uu} N^u$ and, according to our results, N^u is proportional to p_w, it will suffice to demand that p_w vanish. It is not difficult to check that the condition $p_w = 0$ is second class with the constraint $\tilde{\mathcal{H}} = 0$. On the other hand, the stability of our gauge fixing (i.e., $\dot{p}_w = 0$) implies that $[\ln(\underset{\sim}{N} r r')]' = -e^{2w} G/r'$, where

$$G = \frac{1}{2r^3}\left[(c_\theta + c_z v)^2 + c_z^2 r^2 e^{2y}\right]. \quad (2.11)$$

The above differential equation provides then a unique value for $\underset{\sim}{N}$ under the condition that the lapse N be asymptotically unity, namely, that $\lim_{u\to\infty} N = 1$.

In addition, the constraint $\tilde{\mathcal{H}} = 0$ leads to a non-linear and inhomogeneous first-order differential equation for w that, in spite of the apparent complication, can be solved exactly. The solution for vanishing p_w is

$$e^{2w} = \frac{(r')^2 \exp\left(\int_{u_0}^{u} H\right)}{(r_0')^2 e^{-2w_0} - \int_{u_0}^{u} d\hat{u}\, r' G \exp\left(\int_{u_0}^{\hat{u}} H\right)}. \quad (2.12)$$

Here, u_0 is a fixed point, used as the end point in all our integrations (which are over the dependence on the coordinate u), $w_0(t) = w(u = u_0, t)$, and

$$H = \frac{2}{rr'}\left[\frac{(ry')^2}{4} + \frac{(v')^2}{4} e^{-2y} + p_y^2 + p_v^2 r^2 e^{2y}\right]. \quad (2.13)$$

Employing this solution, together with the boundary condition $\lim_{u\to\infty} N = 1$, it is not difficult to integrate the differential equation satisfied by the densitized lapse. One obtains

$$\underset{\sim}{N} = A_\infty \frac{r'}{r} e^{-2w} \exp\left(-\int_u^\infty H\right), \quad (2.14)$$

with

$$A_\infty = \frac{e^{w_\infty - y_\infty/2}}{r'_\infty}, \quad (2.15)$$

w_∞, y_∞, and r'_∞ being the limits of w, y, and r', respectively, when $u \to \infty$. It is worth noting that, in order that w be real, the denominator in Eq. (2.12) has to be positive for all real values of u. We will discuss this point in detail in Sec. IV.

In the rest of our discussion, we will fix u_0 at minus infinity and call $w_0(t) = \lim_{u\to-\infty} w(u,t)$. Furthermore, in order to suppress any explicit time dependence in the solution for e^{2w} given above, we will suppose that w_0 is actually constant. The assumption $\dot{w}_0 = 0$ introduces then some consistency conditions in our system. Employing the fact that $\int du \underset{\sim}{N} \tilde{\mathcal{H}}$ generates the time evolution before one performs the gauge fixing discussed in this subsection, one can see that, on our gauge section,

$$\frac{d(e^{2w})}{dt} = 2A_\infty \exp\left(-\int_u^\infty H\right)(p_v v' + p_y y'). \quad (2.16)$$

Let us now suppose that $\dot{w}_0(t) = \lim_{u\to-\infty} \dot{w}(u,t)$. This commutation of the limit $u \to -\infty$ and the time derivative should occur at least for sufficiently smooth solutions $w(u,t)$ if no material sources are present at minus infinity that might invalidate the vacuum equation of motion for w. Besides, let us admit that A_∞ is finite and that, in the sector of phase space under consideration, H is integrable over the real line. Then, the requirement that w_0 be constant implies that

$$\lim_{u\to-\infty}(p_v v' + p_y y') = 0. \quad (2.17)$$

In the following, we assume that this condition is satisfied. Actually, we will see in Sec. V and in the Appendix that, at least in certain situations, Eq. (2.17) is satisfied once one imposes suitable boundary conditions on the physical degrees of freedom of the system.

Finally, after the gauge fixing explained here, the symplectic form induced on phase space is

$$\Omega_3 = \int_{I\!R} du\,(\mathbf{d}p_v \wedge \mathbf{v} + \mathbf{d}p_y \wedge \mathbf{d}y). \quad (2.18)$$

The system is free of constraints and its physical degrees of freedom are the canonically conjugate pairs of fields (v, p_v) and (y, p_y).

III. REDUCED MODEL

In this section, we will study the constraint-free system obtained with our gauge fixing. We first show in Sec. III A that there exists a reduced Hamiltonian that (at least formally) generates the time evolution. The explicit form of the spacetime metric in terms of the physical degrees of freedom is given in Sec. III B. There, we employ the function $r(u)$ as a radial coordinate, instead of the spatial coordinate u. We also obtain the dynamical equations for the reduced model and show that, in order to eliminate a physical ambiguity coming from the freedom in the choice of origin for y, one can fix the value of y_∞ equal to zero.

A. Reduced Hamiltonian

The equations of motion satisfied by the physical degrees of freedom of our model can be deduced by recalling that, before fixing the gauge associated with the Hamiltonian constraint, the dynamics is generated by the Hamiltonian $\int du \underset{\sim}{N} \tilde{\mathcal{H}}$ via the Poisson brackets determined by the symplectic form (2.10). Once the time derivatives of v, y, and their momenta have been computed in this way, one can evaluate them at $p_w = 0$ and substitute the values of w and $\underset{\sim}{N}$ given in Eqs. (2.12) and (2.14). The results are the dynamical equations that dictate the evolution in our reduced system. Remarkably, it turns out that such equations can be directly obtained in the constraint-free system, endowed with the bracket structure provided by the symplectic form Ω_3, if one employs as reduced Hamiltonian the following function on phase space:

$$H_R = -r'_\infty e^{-w_\infty - y_\infty/2} + \text{ const.} \tag{3.1}$$

Note that, assuming that y_∞ is fixed by the boundary conditions, the only phase-space dependence of H_R is through w_∞, which is obtained from expression (2.12) in the limit that $u \to \infty$. As for the additive constant appearing in Eq. (3.1), it seems natural to fix it so that the Hamiltonian of flat Minkowski spacetime vanishes. We will come back to this point later in this section.

The simplest way to show that H_R provides a reduced Hamiltonian is to check that it leads to the correct equations of motion. Actually, this can be done after a lengthy but trivial calculation. It is important to remember that, in the constraint-free system, all degrees of freedom commute under Poisson brackets with w_0, because we have supposed that this quantity is a fixed constant. Had we not imposed this restriction, w_0 could have contained a non-trivial phase-space dependence.

An alternative proof that H_R is the Hamiltonian that generates the reduced dynamics is the following. Let us call χ_1 the densitized Hamiltonian constraint and χ_2 the gauge-fixing condition $p_w = 0$, and let $c^{(lm)}(u, \bar{u})$ be the matrix that satisfies

$$\int_{I\!R} d\hat{u}\, c^{(lm)}(u,\hat{u})\{\chi_m(\hat{u}),\chi_n(\bar{u})\}_P = \delta^l_n \delta(u-\bar{u})$$
$$= \int_{I\!R} d\hat{u}\, \{\chi_n(u),\chi_m(\hat{u})\}_P\, c^{(ml)}(\hat{u},\bar{u}). \tag{3.2}$$

Here, the Poisson brackets $\{\ ,\ \}_P$ are those corresponding to the symplectic structure (2.10), i.e., before our choice of gauge for the Hamiltonian constraint. The indices l, m, and n, on the other hand, can take the values 1 or 2. Then, after completing the gauge fixing, the brackets of the physical degrees of freedom $\{\xi\} \equiv \{v, p_v, y, p_y\}$ with w are [25]

$$\{\xi(u), w(\bar{u})\} = -\int_{I\!R} d\hat{u} \int_{I\!R} d\check{u}\, \{\xi(u), \chi_1(\hat{u})\}_P\, c^{(1m)}(\hat{u},\check{u}) \times \{\chi_m(\check{u}), w(\bar{u})\}_P, \tag{3.3}$$

where we have used $\{\xi, w\}_P = \{\xi, p_w\}_P = 0$.

The right-hand side of the above formula must be evaluated on our gauge section once all Poisson brackets have been computed. On that section, Eq. (3.2) is solved by the matrix $c^{(11)} = c^{(22)} = 0$ and

$$c^{(12)}(u,\bar{u}) = -c^{(21)}(\bar{u},u) = \frac{\Theta(\bar{u}-u)\underset{\sim}{N}(u)}{\underset{\sim}{N}(\bar{u})r(\bar{u})r'(\bar{u})}, \tag{3.4}$$

up to the addition of a function of time to the Heaviside function $\Theta(\bar{u} - u)$. Such an arbitrary function of time is set in fact equal to zero by the condition that w_0 commute with the physical degrees of freedom, namely, that the right-hand side of Eq. (3.3) vanish in the limit $\bar{u} \to -\infty$. Remember that the Heaviside function $\Theta(x)$ is unity for $x > 0$ and zero otherwise, and that the densitized lapse is given by expression (2.14). Substituting the above value for $c^{(12)}$ in Eq. (3.3), taking the limit $\bar{u} \to \infty$ and recalling that $\underset{\sim}{N} r r'$ tends to $e^{-w_\infty - y_\infty/2} r'_\infty$, whereas

$$\dot{\xi}(u) = \int_{I\!R} d\hat{u}\, \underset{\sim}{N}(\hat{u})\, \{\xi(u), \tilde{\mathcal{H}}(\hat{u})\}_P, \tag{3.5}$$

we arrive at the desired result $\dot{\xi} = \{\xi, H_R\}$. In doing so, we have also used the fact that r'_∞ is a constant given by our gauge fixing and assumed that y_∞ is fixed as a boundary condition.

Taking into account that, apart from a fixed factor, e^{-w_∞} generates the reduced dynamics and that w, determined by expression (2.12), is explicitly time independent, we also see that the quantity w_∞ is in fact a constant of motion: its value remains constant in the classical evolution, although it may vary from one classical solution to another. Of course, the same result applies to the reduced Hamiltonian H_R, whose value is thus conserved by the dynamics of the reduced system.

In arriving at this result, the fact that w_0 can be set equal to a fixed constant plays a fundamental role: otherwise, w_∞ would generally display a non-trivial explicit dependence on time. Remember that, in the absence of

external sources that could affect the value of $\dot{w}$ at minus infinity, the assumption that w_0 is constant (and w smooth) amounts to condition (2.17). In a similar way, assuming that there exist no external sources at infinity that could modify the value of $\dot{w}$ when $u \to \infty$, the constancy of w_∞ turns out to introduce an additional requirement in our system. Arguments like those presented when deducing Eq. (2.17) lead to the conclusion

$$\lim_{u\to\infty} (p_v v' + p_y y') = 0. \tag{3.6}$$

As happens to be the case with the analogous condition at minus infinity, it is seen in Sec. V (and in the Appendix) that, at least in certain situations, this requirement is satisfied as a consequence of the boundary conditions imposed on the physical fields.

B. Metric and equations of motion

Let us now summarize the results obtained so far, but performing a change of coordinates from u to the positive, strictly increasing function r introduced in Sec. II C. Notice that, since the new coordinate r is positive and equal to the determinant of the metric on Killing orbits, it is possible to interpret it as a radial coordinate. We will denote the limits of $r(u)$ when u tends to minus and plus infinity, respectively, by r_0 and r_∞. Obviously, we have $0 \leq r_0 < r_\infty$ and the range of r is (r_0, r_∞). The axis $r = 0$ is in principle excluded from our manifold. The phase space of the reduced model has the symplectic form

$$\bar{\Omega} = \int_{r_0}^{r_\infty} dr\, (\mathbf{d}P_v \wedge \mathbf{d}v + \mathbf{d}P_y \wedge \mathbf{d}y), \tag{3.7}$$

where $P_v = p_v/r'$ and $P_y = p_y/r'$. Thus, the system has four physical degrees of freedom, which are given by the canonical fields $\{v, P_v, y, P_y\}$.

From our discussion in Sec. II, the spacetime metric can be expressed in terms of these fields as

$$\begin{aligned} ds^2 = {} & e^{2\bar{w}+y}\left[-\bar{N}^2 dt^2 + dr^2\right] + e^y r^2 (d\theta + N^\theta dt)^2 \\ & + e^{-y}\left[dz - v d\theta + (N^z - vN^\theta)dt\right]^2 . \end{aligned} \tag{3.8}$$

Here

$$e^{2\bar{w}} = \frac{e^{2w}}{(r')^2} = \frac{E[r]}{E[r_0]e^{-2\bar{w}_0} - \int_{r_0}^{r} d\hat{r}\, G E[\hat{r}]}, \tag{3.9}$$

$$E[r] = \exp\left(-\int_r^{r_\infty} \bar{H}\right), \tag{3.10}$$

$$\bar{H} = \frac{2}{r}\left[\frac{(r\partial_r y)^2}{4} + \frac{(\partial_r v)^2}{4} e^{-2y} + P_y^2 + P_v^2 r^2 e^{2y}\right], \tag{3.11}$$

$$\bar{N} = A_\infty e^{-2\bar{w}} E[r], \tag{3.12}$$

with $A_\infty = e^{\bar{w}_\infty - y_\infty/2}$, $e^{-\bar{w}_0} = r_0' e^{-w_0}$, and G being defined in Eq. (2.11). All integrals are over the dependence on the radial coordinate r and ∂_r denotes the partial derivative with respect to r. In addition, the shift vector is given by

$$\begin{aligned} N^z &= A_\infty \int_r^{r_\infty} \frac{d\hat{r}}{\hat{r}^3} \left(c_\theta v + c_z v^2 + c_z \hat{r}^2 e^{2y}\right) E[\hat{r}], \\ N^\theta &= A_\infty \int_r^{r_\infty} \frac{d\hat{r}}{\hat{r}^3} \left(c_\theta + v c_z\right) E[\hat{r}]. \end{aligned} \tag{3.13}$$

The equations of motion that dictate the dynamics of our reduced system can be deduced, e.g., from Eq. (3.5). One finds

$$\begin{aligned} \dot{v} &= 2A_\infty P_v r e^{2y-2\bar{w}} E[r], \\ \dot{y} &= 2A_\infty \frac{P_y}{r} e^{-2\bar{w}} E[r], \\ \dot{P}_v &= A_\infty \partial_r \left(\frac{\partial_r v}{2r} e^{-2y-2\bar{w}} E[r]\right) - A_\infty \frac{c_z}{2r^3}(v c_z + c_\theta) E[r], \\ \dot{P}_y &= A_\infty \partial_r \left(\frac{\partial_r y}{2} r e^{-2\bar{w}} E[r]\right) - \frac{A_\infty}{2r} e^{-2\bar{w}} E[r] \\ & \quad \times \left[c_z^2 e^{2y+2\bar{w}} + 4P_v^2 r^2 e^{2y} - (\partial_r v)^2 e^{-2y}\right]. \end{aligned} \tag{3.14}$$

On the other hand, in the absence of sources that could modify the time variation of $\bar{w}$ at the end points of the domain of definition of r, the requirement that $\bar{w}$ remain constant at those points [or, strictly speaking, that $e^{2\bar{w}}$ does; see Eq. (2.16)] leads to the condition

$$\lim_{r\to r_0,\, r_\infty} (P_v \partial_r v + P_y \partial_r y) = 0, \tag{3.15}$$

which is the analogue of Eqs. (2.17) and (3.6).

From the above equations of motion, it is easy to see that the Minkowskian solution with boundary condition $\lim_{r\to r_\infty} y = y_\infty$ is obtained by setting $v = P_v = P_y = 0$ and $y = y_\infty$ when the parameters c_z, c_θ, and $\bar{w}_0$ vanish. Using then this flat spacetime as the solution with respect to which one measures the value of the reduced Hamiltonian, one gets $H_R = e^{-y_\infty/2}(1 - e^{-\bar{w}_\infty})$.

Several comments are in order at this stage of our discussion. First, we remark that Minkowski spacetime is a solution of our reduced system only if the constants c_z, c_θ and $\bar{w}_0$ are equal to zero. These constants are supposed to be parameters of the system, and not physical degrees of freedom. So, strictly speaking, Minkowski spacetime cannot be considered a background solution unless the above parameters vanish in our model. Nevertheless, we can always decide to measure the value of the reduced Hamiltonian as referred to its Minkowskian value. What we are doing in this way is to employ a universal reference for all of the reduced models that are obtained with different choices of the parameters c_z, c_θ, and $\bar{w}_0$.

Second, we note that, with the boundary condition $\lim_{r\to r_\infty} y = y_\infty$, the asymptotic norm of the Killing field ∂_z generally differs from unity. The normalized translational Killing field is given by $e^{y_\infty/2}\partial_z$ instead. This fact must be taken into account if one wants to define quantities per asymptotic unit length in the axis direction (such

as, e.g., a linear energy density). In fact, it suffices to redefine the system of units so that $4Ge^{y_\infty/2}=1$, where G is the effective gravitational constant introduced in Sec. II. One can check that, for all practical purposes, the only important consequence of this redefinition is the introduction of a shift in the origin of y that makes the boundary value y_∞ equal to zero. An equivalent way to see that the value taken by y_∞ is physically irrelevant, so that it can be set to vanish, is the following. It is not difficult to check from the expression of the metric that an additive constant in the field y can always be absorbed by a constant scaling of the coordinates z and r, the fields v and P_v, and the constants c_z and c_θ. All four-geometries related by a shift in y and these scaling transformations are thus equivalent. In order to eliminate this redundancy, one can then simply fix the value of y at r_∞. For convenience, we will hence take

$$y_\infty = 0, \qquad A_\infty = e^{\bar{w}_\infty}. \tag{3.16}$$

So the Hamiltonian that generates the dynamics of the reduced model can now be written in the form

$$H_R = 1 - e^{-\bar{w}_\infty}. \tag{3.17}$$

Finally, we notice that, when the constants c_z, c_θ, and $\bar{w}_0$ vanish, the formulas given above for the spacetime metric and reduced Hamiltonian reproduce the results obtained in the literature for cylindrical waves that are regular at the axis [10,23,26].

IV. ENERGY BOUNDS

Let us now show that the linear energy density contained in our system, which is given by the value of the reduced Hamiltonian, is bounded both from above and below, like in the case with regular axis [7]. In doing this, we will only assume that the Hamiltonian formalism is well defined in our reduced model.

In the real phase space of our model, the spacetime metric (3.8) describes the 3+1 decomposition of a Lorentzian spacetime with time coordinate t if and only if $\bar{w}$ [given by Eq. (3.9)] is real. In particular, the reduced Hamiltonian will not generate time evolution unless $\bar{w}_\infty$ is real. So $e^{-\bar{w}_\infty}$ must be strictly positive. As a consequence, we conclude that the Hamiltonian H_R is bounded from above by unity, $H_R<1$. Here, we have ruled out the possibility $e^{-\bar{w}_\infty}=0$ by requiring that metric (3.8) be well defined.

In order to find a lower bound for the Hamiltonian, let us first note that the quantities G and $\bar{H}$ that enter the expression of $\bar{w}$ are positive functions on the real phase space, as can be easily seen from their definitions, taking into account that the radial coordinate r is positive. It is then straightforward to check that $e^{2\bar{w}}$ is a strictly increasing function of r, provided that $\bar{w}$ is actually real. Obviously, this implies that $\bar{w}_\infty \geq \bar{w}_0$. Therefore, for each fixed value of the constant parameter $\bar{w}_0$, the reduced Hamiltonian is also bounded from below: $H_R \geq (1-e^{-\bar{w}_0})$.

When the axis of symmetry is regular, so that $\bar{w}_0$ vanishes, we recover the result $1 > H_R \geq 0$ [7]. Furthermore, assuming as a boundary condition (see Sec. V and the Appendix for a detailed discussion) that v is much smaller than the unit function in the limit that r tends to r_∞ and remembering that the shift vector vanishes asymptotically, one can see that, in this asymptotic region, metric (3.8) describes a conical geometry with deficit angle equal to $2\pi H_R$ (and possibly non-zero angular momentum). Imposing that the deficit angle be positive amounts thus to demanding positivity of the energy. From our discussion above, this positivity could be ensured, e.g., by restricting the constant parameter $\bar{w}_0$ to be non-negative, because then $e^{\bar{w}_\infty} \geq e^{\bar{w}_0} \geq 1$.

Finally, let us notice that, since $e^{2\bar{w}}$ is strictly increasing with r (as far as it is positive) and can be seen to change sign at most once in the positive real axis, the requirement that $\bar{w}$ be real in the domain of definition of r is satisfied if and only if $e^{\bar{w}_\infty}>0$. This last condition is stable under dynamical evolution, because $e^{\bar{w}_\infty}$ is a constant of motion. On the other hand, using formula (3.9) and recalling that $e^{\bar{w}_\infty}$ must be finite, we can rewrite the considered condition as

$$E[r_0]e^{-2\bar{w}_0} > \int_{r_0}^{r_\infty} dr\, G\, E[r]. \tag{4.1}$$

The above inequality can be understood as a restriction on the acceptable values of the phase space variables for each fixed value of $\bar{w}_0$. In the case that c_z and c_θ vanish, the inequality is trivially satisfied, because G is then equal to zero.

In conclusion, condition (4.1) implies the reality of the metric function $\bar{w}$ everywhere in spacetime and guarantees that the range of the reduced Hamiltonian is contained in $[1-e^{-\bar{w}_0}, 1)$, regardless of the specific values taken by the parameters c_z and c_θ of the model.

V. CONSISTENCY OF THE FORMALISM AND SPINNING SOLUTIONS

To some extent, the analysis presented in the previous sections is only formal. The emphasis has been put on removing all the gauge freedom and finding the expressions of the reduced metric and Hamiltonian, rather than on proving that such expressions are well defined. Our aim in this section is to show that, at least in certain situations, the reduced formalism that we have discussed is in fact fully consistent.

Let us summarize the conditions that are necessary for the consistency of the model. First, the metric expression (3.8) must be meaningful everywhere. This implies that the integral

$$I[r] = \int_{r_0}^{r} d\hat{r}\, G\, E[\hat{r}] \tag{5.1}$$

and those that appear in $E[r]$, N^z, and N^θ must converge for all $r \in (r_0, r_\infty)$. In addition, one must demand that $I[r_\infty]$ be finite, so that $e^{2\bar{w}_\infty}$ is well defined. Remember also that this constant of motion has to be positive if the induced metric is positive definite and the reduced Hamiltonian real. This last condition is equivalent to requirement (4.1), which can be interpreted as a dynamically stable restriction on phase space and implies, in particular, that $E[r_0] > 0$. On the other hand, in order for the Hamiltonian formalism to be well defined, the reduced Hamiltonian must not only be real and finite, but also differentiable on phase space, i.e., with respect to variations of the fields v, P_v, y, and P_y. Other consistency conditions that must be satisfied are those given in Eq. (3.15) (which guarantee that $\bar{w}$ is constant at r_0 and r_∞) and that y_∞ can be kept equal to zero. Finally, note that these requirements must hold at all instants of time; i.e., the imposed conditions must be stable.

A. General case

We will first consider the possibility $0 < r_0 < r_\infty < \infty$. Assuming that all fields are sufficiently smooth, the integrals that determine the metric components are then convergent, because they contain no singularities and the interval of integration is bounded. As for the differentiability of the reduced Hamiltonian, a detailed calculation shows that the variation of H_R includes two types of contributions. The first type consists of integrals over $r \in (r_0, r_\infty)$ that converge because of the reasons explained above. The second type are surface terms that arise in the integration by parts of the variations of $\partial_r v$ and $\partial_r y$. These terms must vanish if the Hamiltonian is differentiable. Notice that the derivatives $\partial_r v$ and $\partial_r y$ appear in H_R only via the phase-space dependence of $\bar{H}$, Eq. (3.11), and that there is no functional dependence on $\partial_r P_v$ and $\partial_r P_y$. At least for variations of v and y that are proportional to the Hamiltonian variations $\dot{v}$ and $\dot{y}$ at r_0 and r_∞, a careful analysis proves that the considered surface terms vanish as a consequence of Eqs. (3.15). In this sense, in order to guarantee that the reduced formalism is rigorously defined, one would only need to impose inequality (4.1) at a certain instant of time and conditions (3.15) and $y_\infty = 0$ for all values of t, so that these last requirements are stable.

However, there is no obvious way in which condition $y_\infty = 0$ and Eqs. (3.15) can be satisfied and preserved in the evolution. Of course, one could assume that there exist external sources acting on the boundaries of the spacetime that invalidate the applicability of the reduced equations of motion (3.14) at $r = r_0$ and $r = r_\infty$. The possibility of regaining consistency in this way will not be explored here. There still exists another situation in which our consistency conditions can be satisfied, namely, in solutions whose fields v and y are constant (both with respect to t and r) outside a certain region of the form $r \in (r_1, r_2)$, where $r_0 < r_1 \leq r_2 < r_\infty$. Of course, we require that the value of y_∞ vanish. Using the dynamical equations (3.14), one can easily construct solutions of this type provided that the constant c_z is equal to zero: it suffices to assume that the momenta vanish for r in $(r_0, r_1] \cup [r_2, r_\infty)$. Note, nevertheless, that such solutions can always be extended to the whole region $r \in (0, \infty)$ by keeping the fields $\{v, P_v, y, P_y\}$ constant outside (r_1, r_2).

We are thus naturally led to consider the case $r_0 = 0$ and $r_\infty = \infty$, either because otherwise we cannot ensure the consistency of the formalism or because the only interesting solutions when r has a bounded domain of definition can be trivially extended to the semiaxis $(0, \infty)$. We will first analyze models with non-vanishing parameter c_z. The discussion for $c_z = 0$ will be presented in the next subsection.

When $c_z \neq 0$, the conditions that $I[\infty]$ be finite and $E[0]$ positive imply

$$\infty > \frac{I[\infty]}{E[0]} \geq \int_0^\infty \frac{dr}{2r^3} c_z^2 \left[\left(v + \frac{c_\theta}{c_z} \right)^2 + r^2 e^{2y} \right]. \quad (5.2)$$

In the last inequality, we have used that $E[r]$ increases with r, since $\bar{H}$ is a positive function on phase space. The convergence of the last integral would require that

$$\lim_{r \to 0, \infty} e^y = \lim_{r \to 0, \infty} \frac{c_z v + c_\theta}{r} = 0, \quad (5.3)$$

where the limits are taken both at zero and at infinity. Clearly, the first of these conditions cannot be satisfied (remember, in particular, that we have assumed $y_\infty = 0$). This means that there exist divergences in the metric functions when c_z does not vanish. More explicitly, the denominator in $e^{2\bar{w}}$ diverges. Furthermore, it is not difficult to realize that the divergent terms in $I[r]/E[r_0]$ when $r_0 \to 0$ cannot actually be absorbed by a kind of renormalization of the constant parameter $e^{-2\bar{w}_0}$ that appears in Eq. (3.9), because those terms depend on the behavior of the fields v and y around the axis $r = 0$ and vary, in general, from one solution to another.

B. Case $c_z = 0$

Let us now consider the only remaining possibility, i.e., the case in which the domain of definition of r is the whole semiaxis $(0, \infty)$ and the constant parameter c_z vanishes. Using Eq. (2.5) and a line of reasoning similar to that discussed in Refs. [7,18], it is easy to show that $2\Pi^u_a = c_a/2$ (where we have used the notation of Sec. II A) is precisely the value of the surface term that must be added to the smeared momentum constraint $\int du N^i H_i$ in order to make it differentiable on phase space when the shift vector N^i equals δ^i_a ($a = 1$ or 2) in the asymptotic region $u \gg 1$ and vanishes for $u \ll -1$. Therefore, with our choice of units, $c_\theta/2$ and $c_z/2$ are the values per unit

length of the angular momentum and the linear momentum in the z direction, respectively. The solutions that we are going to study can thus be regarded as spacetimes with a possibly singular axis of symmetry, a vanishing linear momentum in the direction of this axis and, in general, a non-zero angular momentum.

If we now analyze the behavior of $I[r]$ when r_0 approaches the origin, we easily see that this integral still diverges, like when c_z differed from zero. However, the leading term in $I[r]/E[r_0]$ is now the same for all solutions in our model and can be absorbed in the denominator of $e^{2\bar{w}}$ by a renormalization of the constant $e^{-2\bar{w}_0}$. Moreover, if one assumes boundary conditions such that

$$\lim_{r\to 0}\frac{\bar{H}}{r^{1+\epsilon}}=0 \tag{5.4}$$

for a certain number $\epsilon>0$, one can check that the only divergent term in $I[r]/E[r_0]$ when r_0 tends to zero has the form $c_\theta^2/(4r_0^2)$ and is thus universal. So this term can be removed by redefining $e^{-2\bar{w}_0}=D+c_\theta^2/(4r_0^2)$ and taking the limit $r_0\to 0$, where D is a constant parameter. Expression (3.9) can then be rewritten

$$e^{2\bar{w}}=\frac{4\bar{E}[r]r^2}{c_\theta^2+4Dr^2-2c_\theta^2 r^2\int_0^r d\hat{r}\,\hat{r}^{-3}\left(\bar{E}[\hat{r}]-1\right)}, \tag{5.5}$$

$$\bar{E}[r]=\frac{E[r]}{E[0]}=\exp\left(\int_0^r\bar{H}\right). \tag{5.6}$$

Several comments are in order at this point. First, notice that $\bar{E}[r]$ is a strictly increasing function of r that is always equal or greater than unity, because $\bar{H}$ is positive. In addition, assuming that the fields v, P_v, y, and P_v are sufficiently smooth in the region $0<r<\infty$, condition (5.4) guarantees that the integrals appearing in $\bar{E}[r]$ and in the denominator of $e^{2\bar{w}}$ are well defined and convergent for $r\in[0,\infty)$. On the other hand, the condition that $E[0]$ be positive amounts to requiring that $\bar{E}[\infty]$ be finite. This is ensured, e.g., when there exists a strictly positive number $\delta>0$ such that

$$\lim_{r\to\infty} r^{1+\delta}\bar{H}=0. \tag{5.7}$$

Imposing this asymptotic behavior, it is straightforward to check that $e^{2\bar{w}_\infty}$ is well defined and positive if

$$D>c_\theta^2\int_0^\infty\frac{dr}{2r^3}(\bar{E}[r]-1). \tag{5.8}$$

This requirement replaces Eq. (4.1), owing to the redefinition of $\bar{w}_0$. In particular, since $\bar{E}[r]\geq 1$, it is necessary that D be positive. We will thus restrict our discussion to the case $D>0$ from now on. Note also that, since $e^{2\bar{w}_\infty}$ is a constant of motion, inequality (5.8) is preserved in the evolution.

Concerning the shift vector, the integrals in Eq. (3.13) are meaningful for all $r\in(0,\infty)$ if v has a finite limit at infinity, which we will assume to vanish (so that the asymptotic metric describes a conical geometry with possibly a non-zero angular momentum). Hence, all metric functions are well defined everywhere in spacetime. Furthermore, from Eqs. (5.4) and (3.11), it follows that $\partial_r v$ is much smaller than $r^{1+\epsilon/2}$ close to the origin. Then, supposing that v is smooth enough in that region, it is not difficult to check that N^z-vN^θ has a finite limit when $r\to 0$. On the other hand, we can rewrite N^θ as

$$N^\theta=e^{\bar{w}_\infty}E[0]c_\theta\left\{\frac{1}{2r^2}+\int_r^\infty\frac{d\hat{r}}{\hat{r}^3}(\bar{E}[\hat{r}]-1)\right\}, \tag{5.9}$$

so that this shift component diverges at the axis $r=0$. From expressions (3.8) and (3.12) we then see that the only potentially divergent terms in the four-metric when $r\to 0$ are included in the diagonal t component, and are given by

$$-\,e^y\left\{e^{2\bar{w}_\infty}e^{-2\bar{w}}(E[r])^2-r^2(N^\theta)^2\right\}. \tag{5.10}$$

Nevertheless, taking into account that condition (5.4) guarantees that $E[r]-E[0]$ is much smaller than $r^{2+\epsilon}$ for $r\to 0$, one can check that the value obtained from Eqs. (5.5) and (5.9) for these presumably divergent terms is in fact finite when r vanishes. Thus, in our coordinate system, the metric components are well defined even in the limit in which one reaches the axis of symmetry.

So far we have already proved that, when $c_z=0$ and $c_\theta\neq 0$, there is no problem with the expressions of the metric and the reduced Hamiltonian, assuming that conditions (5.4), (5.7), and (5.8) are satisfied and v and y vanish at infinity. In addition, the consistency of our formalism implies Eqs. (3.15) and the differentiability of the reduced Hamiltonian. Finally, all these conditions must be stable. In the Appendix, we present boundary conditions on the fields $\{v,P_v,y,P_y\}$ at the axis and at infinity that ensure that all these requirements are satisfied. Nonetheless, in order to demonstrate the relevance of the reduced model, it actually suffices to show that the set of acceptable solutions is infinite dimensional.

In fact, this last statement can be easily proved. As we have already commented, when $c_z=0$, the dynamical equations (3.14) admit sufficiently smooth (even C^∞) solutions in which all fields are constant outside a bounded interval for r of the form (r_1,r_2), where r_1 and r_2 satisfy $0<r_1\leq r_2<\infty$ but are otherwise arbitrary. Besides, outside (r_1,r_2) the momenta P_v and P_y vanish. The fields v and y are set equal to zero in the interval $[r_2,\infty)$, so that the condition that these fields vanish asymptotically is satisfied. To avoid topological complications on the sections of constant time in the neighborhood of the axis, we will also assume that v vanishes in the region $r\in(0,r_1]$. Finally, y will take a constant, finite value y_0 in that region. For this infinite family of solutions it is straightforward to see that all the conditions necessary for consistency are satisfied, including stability, except maybe the differentiability of the reduced Hamiltonian and Eq. (5.8). The differentiability of the Hamiltonian can be checked following a line of reasoning similar to

that explained in the beginning of Sec. V A. The surface terms that appear in the variation of the Hamiltonian vanish because so do $\{\partial_r v, P_v, \partial_r y, P_y\}$ at the axis and at infinity. The remaining terms are integrals that converge because they get no contribution outside the bounded region (r_1, r_2), where all integrands are sufficiently smooth. Hence, the variation is well defined and the reduced Hamiltonian differentiable. We are only left with condition (5.8), which must be regarded as a restriction on phase space which is satisfied always in the evolution if so is at a single instant of time.

It is easy to see that, for any strictly positive constant D, there exists an infinite dimensional set of initial values for our fields $\{v, P_v, y, P_y\}$ such that inequality (5.8) holds. Actually, the minimum of the right-hand side in that inequality is just zero and is reached when $\bar{H}$ vanishes. Given expression (3.11) and that $y_\infty = v_\infty = 0$, this occurs only for the solution with vanishing fields. This flat solution can be taken as a background for our model with fixed parameters c_θ and $D > 0$. The background metric adopts the expression

$$ds^2 = -dt^2 + \frac{c_\theta}{\sqrt{D}} dt d\theta + r^2 d\theta^2 + dz^2 + \frac{4r^2 dr^2}{c_\theta^2 + 4Dr^2}, \quad (5.11)$$

which is precisely the line element originated by a spinning cosmic string, restricted to the region where causality is preserved and there exist no closed timelike curves (CTC's) [16,17]. A more familiar form for this metric, which can be continued to the region $-c_\theta^2/(4D) < r^2 \leq 0$ at the cost of introducing CTC's, is obtained with the change of coordinate $D\rho^2 = r^2 + c_\theta^2/(4D)$. Condition (5.8) is clearly satisfied by our background solution, and one can check that it is satisfied as well at least in a certain region of phase space around the origin $v = P_v = y = P_v = 0$. Therefore, the set of admissible spinning solutions is infinite dimensional.

Finally, let us note that, when $c_z = 0$, the lower bound obtained for the reduced Hamiltonian in Sec. IV can be improved. From Eq. (5.5) and the fact that $\bar{E}[r] \geq 1$, one gets $e^{-2\bar{w}_\infty} \leq D$. We then conclude that the value of the reduced Hamiltonian, which provides the energy per unit length in the axis direction, is always contained in the interval $[1 - \sqrt{D}, 1)$. On the other hand, as we have commented on, metric (3.8) describes in the asymptotic region a conical geometry with angular momentum proportional to c_θ and deficit angle equal to $2\pi H_R$. Hence, positivity of the energy and the deficit angle can be ensured, e.g., by simply restricting the parameter D so that $1 \geq D > 0$.

VI. CONCLUSIONS AND FURTHER COMMENTS

We have proposed a gauge-fixing procedure that removes all the non-physical degrees of freedom in vacuum cylindrical spacetimes. Our definition of cylindrical symmetry is less restrictive than that usually employed in the literature, in the sense that we have assumed the existence of two commuting spacelike Killing fields, one of them rotational and the other one translational, but we have not imposed the condition that the spacetime contain the axis of rotational symmetry, namely, the set of points where the metric on Killing orbits degenerates. This relaxation of the conditions for cylindrical symmetry has allowed us to include in our discussion spacetimes whose Killing orbits are not surface orthogonal, so that the line element cannot be written, in general, in block-diagonal form using two-metrics. The price to be paid for this generalization is that now the axis of symmetry, which is located in principle outside the manifold, may actually be singular and contain linear sources.

Our gauge fixing leads to a reduced midisuperspace model that is totally free of constraints and depends on three constant parameters. Two of these parameters, namely, c_z and c_θ, determine, respectively, the constant values of the linear momentum in the axis direction and the angular momentum of the system. The third parameter, $\bar{w}_0$, is the fixed limit when one approaches the axis $(r \to r_0)$ of $\bar{w}$, the metric function that appears in the radial component of the line element (3.8). The phase space of the reduced model is infinite dimensional and can be described using the set of canonically conjugate fields $\{v, P_v, y, P_y\}$. We have obtained the general expression of the four-metric in terms of these physical degrees of freedom and found the equations of motion that govern the evolution of these independent fields. Moreover, we have proved that the dynamics of the model is in fact generated by a reduced Hamiltonian, given by $1 - e^{-\bar{w}_\infty}$. Here, $\bar{w}_\infty$ is the limit of the metric function $\bar{w}$ at large distances from the axis $(r \to r_\infty)$. The value of this Hamiltonian is a constant of motion that provides the amount of energy that is present in the system per unit length in the axis direction. The origin of energy has been chosen to vanish for flat, Minkowski spacetime.

One might wonder whether the expression of the reduced Hamiltonian could also have been obtained from the Hilbert-Einstein action supplemented with boundary terms via a reduction process. Actually, the answer turns out to be in the affirmative, but only if the surface terms are suitably chosen. One can start with the Hamiltonian form of the gravitational action corrected with the standard surface terms that appear when the manifold has a timelike boundary [27]. In our case, this boundary consists of two disconnected parts: an internal boundary at $r = r_0$ and an external one at $r = r_\infty$ (if necessary, one can take the limits $r_0 \to 0$ and $r_\infty \to \infty$ after completing all calculations). It is then possible to show that, if one only includes the surface terms that correspond to the external boundary, the reduction explained in Sec. II leads to

$$S_R = \int dt \left[e^{-\bar{w}_\infty} - 1 + \int_{r_0}^{r_\infty} dr (P_v \dot{v} + P_y \dot{y}) \right], \quad (6.1)$$

which is in fact the action expected for the reduced system. The integral over r determines the sympletic structure, whereas the other factors provide the reduced Hamiltonian. Note that we have normalized the action so that it vanishes for Minkowski spacetime. In the case that the axis of symmetry is regular, which happens only if c_z, c_θ, and $\bar{w}_0$ vanish, the surface terms at $r_0 = 0$ that have been obviated are in fact spurious, because the internal boundary does not exist. But in the general, singular case, we really need to exclude those surface corrections in order to arrive at the correct reduced action. Since the action obtained after reduction depends on the choice of boundary terms, it is clear that the symmetric criticality principle does not generally hold in the system [24].

We have also analyzed in detail the conditions that guarantee that the reduced formalism is consistent. In particular, we have discussed under what circumstances the metric expressions are always well defined and the reduced Hamiltonian is real, finite and differentiable on phase space. In addition, we have checked whether one can safely impose that, at all instants of time, the fields y and v vanish asymptotically and Eqs. (3.15) hold. These equations are necessary to ensure that the parameter $\bar{w}_0$ and the value of the reduced Hamiltonian are constant. We have proved that, when the radial coordinate r is defined over the whole semiaxis $(0, \infty)$ and there is no linear momentum in the direction of the symmetry axis, all the consistency requirements are satisfied provided that the fields $\{v, P_v, y, P_y\}$ are subject to appropriate boundary conditions. We have then particularized our study to models with $r \in (0, \infty)$ and $c_z = 0$ but, in general, with non-vanishing angular momentum, $c_\theta \neq 0$.

For such models, the only apparent problem is a divergence in the denominator of $e^{2\bar{w}}$ in Eq. (3.9) when $r_0 \to 0$. We have shown, however, that this divergence can be absorbed by a redefinition of the constant $e^{-2\bar{w}_0}$. We have called D the renormalized constant, which must be strictly positive. After this redefinition of parameters, the metric functions are not only well defined everywhere in spacetime; in addition, with our choice of coordinates, all metric components turn out to have a finite limit when the axis of symmetry is approached. Assuming boundary conditions like those given in the Appendix, the reduced formalism is fully consistent. Besides, the reduced Hamiltonian, which determines the linear energy density contained in the system, is then bounded both from above and below, like in the case with a regular axis of symmetry [7]. More explicitly, in each of the models with constant parameters c_θ and D (with $c_z = 0$), the range of the reduced Hamiltonian is included in the semi-open interval $[1 - \sqrt{D}, 1)$. Furthermore, if the deficit angle in the asymptotic region $r \gg 1$ is positive, so must be the energy density per unit length along the axis.

In the models with $r \in (0, \infty)$ and $c_z = 0$ but, possibly, a non-vanishing angular momentum, a particularly interesting set of solutions is provided by the following family. We consider a bounded interval (r_1, r_2), with $0 < r_1 \leq r_2 < \infty$, and fields that, at a certain instant of time $t = t_0$, satisfy the conditions that (1) v, P_v, and P_v vanish outside the region $r \in (r_1, r_2)$, (2) y be constant for r in $(0, r_1]$ and vanish in $[r_2, \infty)$, (3) Eq. (5.8) be satisfied, and (4) the fields be sufficiently smooth as functions of r (let us say C^∞). These requirements on the fields are in fact stable in the evolution. One can then check that all conditions necessary for the consistency of the reduced formalism are satisfied on these solutions.

Note that we can regard the values of $\{v, P_v, y, P_y\}$ at $t = t_0$ just as initial data that can be evolved by integrating (e.g., by numerical methods) the dynamical equations (3.14). As we have commented, the result of this integration is a solution satisfying conditions (1)-(4) at all instants of time. In this way, one can actually obtain an infinite number of solutions whose isometry group is not orthogonally transitive (unless $c_\theta = 0$).

In addition, it is possible to show [28] that, at short distances from the axis, $r \ll 1$, all of these solutions approach the line element corresponding to a spinning cosmic string with angular momentum per unit length equal to $c_\theta/2$ and deficit angle given by $2\pi(1 - \sqrt{D})$. Indeed, the metric of this string in the region where no CTC's are present can be obtained by simply setting the fields $\{v, P_v, y, P_y\}$ equal to zero [see Eq. (5.11)]. As a consequence, one can view the metric of the spinning cosmic string as a flat background for the considered family of solutions in the model with constant values of the parameters c_θ and D.

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APPENDIX

In this appendix, we present suitable boundary conditions for the fields $\{v, P_v, y, P_y\}$ of the reduced model with vanishing parameter c_z and $r \in (0, \infty)$. The proof that these conditions are stable under dynamical evolution and that they ensure the consistency of the reduced Hamiltonian formalism will be given elsewhere [28].

The conditions at infinity, $r \to \infty$, are that the field v vanish and that

$$P_v = O(r^{-1}), \qquad y = O(1), \qquad P_y = O(1).$$

The notation $f = O(g)$ means that there exists a strictly positive number $\varepsilon > 0$ such that the function f is much smaller than $r^{-\varepsilon} g$ in the asymptotic region, i.e., $\lim_{r\to\infty} r^{\varepsilon} f/g = 0$. Note that the above conditions imply, in particular, that $y_\infty = 0$.

On the other hand, if the constant parameter c_θ vanishes, we can impose the following conditions in the vicinity of the axis $r=0$:

$$v = o(r^2), \quad P_v = o(r), \quad y = y_0(t) + o(r^2), \quad P_y = o(r).$$

Here, $y_0(t)$ is a time-dependent function, we have assumed that v vanishes at the axis, and the notation $f = o(g)$ is employed for functions whose quotient f/g has a finite limit when $r \to 0$. Finally, in the case with non-vanishing global angular momentum, $c_\theta \neq 0$, an appropriate behavior around the axis $r=0$ is

$$v = o(r^4), \quad P_v = o(r^5), \quad y = y_0(t) + o(r^6), \quad P_y = o(r^3).$$

[1] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[2] G. Beck, Z. Phys. **33**, 713 (1925).

[3] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).

[4] P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Lit. Mainz Abh. Math. Naturwiss. Kl. **1960** No. 2 (1960); A. S. Kompaneets, Zh. Éksp. Teor. Fiz. **34**, 953 (1958) [Sov. Phys. JETP **7**, 659 (1958)].

[5] K. S. Thorne, Phys. Rev. **138**, B251 (1965).

[6] K. Kuchař, Phys. Rev. D **4**, 955 (1971).

[7] A. Ashtekar and M. Varadarajan, Phys. Rev. D **50**, 4944 (1994).

[8] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D **55**, 669 (1997); **55**, 687 (1997).

[9] M. Varadarajan, Phys. Rev. D **52**, 2020 (1995).

[10] J. D. Romano and C. G. Torre, Phys. Rev. D **53**, 5634 (1996).

[11] J. Carot, J. M. M. Senovilla, and R. Vera, Class. Quantum Grav. **16**, 3025 (1999).

[12] B. Carter, J. Math. Phys. **10**, 70 (1969).

[13] B. K. Berger, P. T. Chruściel, and V. Moncrief, Ann. Phys. (N.Y.) **237**, 322 (1995).

[14] This result can be deduced by a generalization of Papapetrou's theorem from the stationary axi-symmetric case to the cylindrical case. See A. Papapetrou, Ann. Inst. Henri Poincaré Sect. A **4**, 83 (1966); W. Kundt and M. Trümper, Z. Phys. **192**, 419 (1966).

[15] J. R. Gott, Astrophys. J. **288**, 422 (1985); W. A. Hiscock, Phys. Rev. D **31**, 3288 (1985); B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985); A. Vilenkin, Phys. Rep. **121**, 263 (1985); in *Three Hundred Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987).

[16] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).

[17] P. O. Mazur, Phys. Rev. Lett. **57**, 929 (1986); **59**, 2380 (1987); J. Samuel and B. R. Iyer, *ibid.* **59**, 2379 (1987); D. Harari and A. Polychronakos, Phys. Rev. D **38**, 3320 (1988).

[18] M. Henneaux, Phys. Rev. D **29**, 2766 (1984).

[19] S. A. Hayward, Class. Quantum Grav. **17**, 1749 (2000).

[20] G. A. Mena Marugán and M. Montejo, Phys. Rev. D **58**, 104017 (1998).

[21] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[22] R. Geroch, J. Math. Phys. **13**, 394 (1972).

[23] M. E. Angulo and G. A. Mena Marugán, gr-qc/0002056, Int. J. Mod. Phys. D (in press).

[24] C. G. Torre, Int. J. Theor. Phys. **38**, 1081 (1999).

[25] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monograph Series No. 2 (Yeshiva University, New York, 1964).

[26] A. Ashtekar and M. Pierri, J. Math. Phys. **37**, 6250 (1996).

[27] S. W. Hawking and C. J. Hunter, Class. Quantum Grav. **13**, 2735 (1996).

[28] G. A. Mena Marugán and N. Manojlović, *Asymptotic behaviour of cylindrical waves interacting with spinning strings* (in preparation).